

Collective oscillations of a 1D trapped Bose gas

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Starting from the hydrodynamic equations of superfluids, we calculate the frequencies of the collective oscillations of a harmonically trapped Bose gas for various 1D configurations. These include the mean field regime described by Gross-Pitaevskii theory and the beyond mean field regime at small densities described by Lieb-Liniger theory. The relevant combinations of the physical parameters governing the transition between the different regimes are discussed.

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Recent experiments on trapped Bose gases at low temperature have pointed out the occurrence of characteristic 1D features. These include deviations of the aspect ratio and of the release energy [1,2] from the 3D behaviour as well as the appearance of thermal fluctuations of the phase, peculiar of 1D configurations [3]. Interest in 1D interacting Bose gases arises from the occurrence of quantum features which are not encountered in 2D and 3D. For example in 1D the fluctuations of the phase of the order parameter rule out the occurrence of long range order even at zero temperature [4]. Such systems cannot be in general described using traditional mean field theories and require the development of a more advanced many-body approach. In the case of 1D Bose gases interacting with repulsive zero-range forces, this has been implemented by Lieb and Liniger [5] who studied both the equation of state and the spectrum of elementary excitations of a uniform gas. In the presence of harmonic trapping, 1D Bose gases exhibit new interesting features. The corresponding equilibrium properties have been already discussed in a recent series of theoretical papers (see [6–8] and references therein). In the present work we investigate the consequences of harmonic trapping on the collective oscillations of an interacting 1D Bose gas at zero temperature. We will consider various configurations, ranging from the mean field regime [9] where the healing length is larger than the average interparticle distance, to the Tonks-Girardeau limit [10] where the gas acquires Fermi like properties. We will show that the frequency of the lowest compression mode provides a useful indicator of the different regimes.

We start our discussion from the hydrodynamic equations of superfluids in 1D

$$\frac{\partial}{\partial t} \delta n_1 + \frac{\partial}{\partial z} (n_1 v) = 0, \quad (1)$$

$$\frac{\partial}{\partial t} v + \nabla_z \left(\mu_{\ell e}(n_1) + V_{ext} + \frac{1}{2} m v^2 \right) = 0, \quad (2)$$

which describe the dynamic behaviour of such systems at zero temperature. In these equations $n_1(z, t)$ is the 1D density of the gas, $v(z, t)$ is the velocity field, while $V_{ext}(z)$ is the external trapping potential which in the following will be assumed to be harmonic: $V_{ext}(z) = m\omega_z^2 z^2/2$. The hydrodynamic equations of superfluids have been already successfully employed to predict the collective frequencies of 3D trapped Bose-Einstein condensates [11]. A crucial ingredient of these equations is the local equilibrium (ℓe) chemical potential $\mu_{\ell e}$, which should be evaluated for a uniform 1D gas ($V_{ext} = 0$) at the density n_1 . The applicability of the above equations requires the validity of the local density approximation along the z -th direction. This is expected to be accurate for sufficiently large systems. Furthermore, Eqs.(1,2) should be limited to the study of macroscopic phenomena where variations in space take place over distances larger than the average distance between particles. From Eq.(2) one can easily calculate the ground state profile through the equation

$$\mu_{\ell e}(n_1(z)) + V_{ext}(z) = \mu. \quad (3)$$

The collective oscillations are instead determined by writing the density in the form $n_1(z, t) = n_1(z) + e^{-i\omega t} \delta n_1(z)$, with the function $\delta n_1(z)$ obeying the linearized equation

$$\omega^2 \delta n_1(z) = \frac{1}{m} \nabla_z \left[n_1(z) \nabla_z \left(\frac{\partial \mu_{\ell e}}{\partial n_1} \delta n_1(z) \right) \right], \quad (4)$$

which immediately follows from Eqs.(1,2). In the case of uniform systems ($V_{ext} = 0$) one has plane wave solutions $\delta n_1 = \exp(iqz)$ with $\omega^2 = c^2 q^2$, where q is the wave vector of the excitation and $c^2 = n_1(\partial \mu_{\ell e} / \partial n_1) / m$ is the square of the sound velocity. It is worth reminding that the applicability of the hydrodynamic equations is not limited to the mean field scenario. Actually in [5] it has been proven that in 1D Bose gases the velocity of sound, derived from the macroscopic compressibility, coincides with the one derived from the microscopic calculation of the phonon excitation spectrum also in regimes far from mean field.

In the first part of the work we evaluate $\mu_{\ell e}(n_1)$ and the corresponding solutions for the collective oscillations in the framework of the mean field Gross-Pitaevskii theory. Even in this regime one can explore a rich variety of situations ranging from the Thomas-Fermi regime in the radial direction to the one of tight confinement where the motion in the radial direction is frozen. In the second part we extend the analysis to regimes beyond mean field, including the limit of the Tonks-Girardeau gas.

Let us consider a uniform system of length L in the z -th direction and confined by a harmonic potential $V(r_\perp) = m\omega_\perp^2 r_\perp^2/2$ in the radial direction. By writing the order parameter in the form $\Psi = \sqrt{n_1} f(\rho_\perp)/a_\perp$, where $n_1 = N/L$ is the 1D density, $a_\perp = \sqrt{\hbar/m\omega_\perp}$ is the oscillator length in the radial direction and $\rho_\perp = r_\perp/a_\perp$ is the dimensionless radial coordinate, the 3D Gross-Pitaevskii equation yields the dimensionless equation

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial \rho_\perp^2} - \frac{1}{2\rho_\perp}\frac{\partial}{\partial \rho_\perp} + \frac{1}{2}\rho_\perp^2 + 4\pi an_1 f^2\right) f = \frac{\mu_{\ell e}}{\hbar\omega_\perp} f \quad (5)$$

for the function f obeying the normalization condition $2\pi \int |f(\rho_\perp)|^2 \rho_\perp d\rho_\perp = 1$. In Eq.(5), $\mu_{\ell e}/\hbar\omega_\perp$ is the chemical potential in units of the radial quantum oscillator energy. Eq. (5) shows that the relevant dimensionless parameter of the problem is an_1 . It is worth considering two important limits. If $an_1 \gg 1$, one enters the radial Thomas-Fermi regime (hereafter called 3D cigar), where many configurations of the harmonic oscillator Hamiltonian are excited in the radial direction and the equation of state takes the analytic form

$$\frac{\mu_{\ell e}}{\hbar\omega_\perp} = 2(an_1)^{1/2}. \quad (6)$$

Notice that in this limit the chemical potential is not linear in the density. This implies, in particular, that the sound velocity is related to the chemical potential by the law $c^2 = \mu_{\ell e}/2m$ [12] rather than by the Bogoliubov relation $c^2 = \mu_{\ell e}/m$. A second important case is the perturbative regime where $an_1 \ll 1$ (hereafter called 1D mean field). In this case the solution of (5) approaches the Gaussian ground state of the radial harmonic oscillator and one finds the linear law

$$\frac{\mu_{\ell e}}{\hbar\omega_\perp} = 1 + 2an_1 \quad (7)$$

for the chemical potential.

If the gas is harmonically trapped also along the z -th direction, the density n_1 exhibits a z -th dependence which is worth calculating as a function of the relevant parameters of the problem: the scattering length a , the number N of atoms and the trapping frequencies ω_\perp and ω_z . To this purpose one has to solve Eq.(3) by imposing the normalization condition $\int n_1(z) dz = N$. A useful quantity is the Thomas-Fermi radius Z defined by the

value of z at which the equilibrium density $n_1(z)$ vanishes. According to Eq.(3), one has $\mu - \mu_{\ell e}(an_1 = 0) = (1/2)m\omega_z^2 Z^2$. In terms of Z , Eq.(3) can be rewritten as $\tilde{\mu}_{\ell e}(an_1(z)) = (m\omega_z^2 Z^2/2\hbar\omega_\perp)(1 - z^2/Z^2)$, where we have defined the dimensionless quantity $\tilde{\mu}_{\ell e}(an_1(z)) = [\mu_{\ell e}(an_1(z)) - \mu_{\ell e}(an_1 = 0)]/\hbar\omega_\perp$. This function is fixed by the solution of the Gross-Pitaevskii equation (5). Its inverse $\tilde{\mu}_{\ell e}^{-1}$ gives the value of an_1 as a function of z , so that the normalization condition obeyed by the density can be written as

$$\frac{Za_\perp}{a_z^2} \int_{-1}^1 \tilde{\mu}_{\ell e}^{-1} \left[\frac{1}{2} \left(\frac{Za_\perp}{a_z^2} \right)^2 (1 - t^2) \right] dt = \frac{Naa_\perp}{a_z^2}, \quad (8)$$

with $t = z/Z$. Eq. (8) explicitly points out the relevance of the dimensionless combination Naa_\perp/a_z^2 where $a_z = \sqrt{\hbar/m\omega_z}$ is the oscillator length in the axial directions. From Eq.(8) one can calculate, for a given choice of the parameters, the radius Z and hence the 1D density profile. In the 3D cigar limit $Naa_\perp/a_z^2 \gg 1$, one has $Z = (a_z^2/a_\perp)(15Naa_\perp/a_z^2)^{1/5}$ and

$$n_1(z) = \frac{1}{16a} \left(\frac{15Naa_\perp}{a_z^2} \right)^{4/5} \left(1 - \frac{z^2}{Z^2} \right)^2. \quad (9)$$

In the 1D mean field limit $Naa_\perp/a_z^2 \ll 1$, one instead finds [7] $Z = (a_z^2/a_\perp)(3Naa_\perp/a_z^2)^{1/3}$ and

$$n_1(z) = \frac{1}{4a} \left(\frac{3Naa_\perp}{a_z^2} \right)^{2/3} \left(1 - \frac{z^2}{Z^2} \right). \quad (10)$$

The density profiles are different in the two regimes, reflecting the different behaviour of the equation of state. The conditions of applicability of the local density approximation employed above are determined by requiring that $Z \gg a_z$. In the 1D mean field regime this implies the non trivial condition $(a_z/a_\perp)(Naa_\perp/a_z^2)^{1/3} \gg 1$.

Let us now discuss the behaviour of the collective oscillations. The hydrodynamic equation (4) has simple analytic solutions if the density derivative of the chemical potential is a power law function: $\partial\mu_{\ell e}/\partial n_1 \propto n_1^{\gamma-1}$. The Thomas-Fermi and the 1D mean field regimes belong to this class of solutions with $\gamma = 1/2$ and $\gamma = 1$ respectively. Also the Tonks-Girardeau limit (see Eq.(13) below) belongs to the same class with $\gamma = 2$. Hence in these three relevant limits the dispersion relation for the collective frequencies can be obtained analytically. By looking for solutions of the form $n_1^{\gamma-1}\delta n_1(z) = z^k + az^{k-2} \dots$, where $k \leq 1$ and only positive powers of z are included in the polynomial, one finds the result

$$\omega^2 = \omega_z^2 \frac{k}{2} [2 + \gamma(k-1)]. \quad (11)$$

The case $k = 1$ corresponds to the center of mass motion whose frequency is given by $\omega = \omega_z$ independent of the

value of γ . The most interesting $k = 2$ case (lowest compressional mode) is instead sensitive to the regime considered. One finds $\omega^2 = (5/2)\omega_z^2$, $\omega^2 = 3\omega_z^2$ and $\omega^2 = 4\omega_z^2$ for the 3D cigar, 1D mean field and Tonks-Girardeau regimes respectively. The result $\omega^2 = (5/2)\omega_z^2$ was first derived in [11] by solving the hydrodynamic equations for a trapped 3D system in the limit of a highly elongated trap ($\omega_z \ll \omega_\perp$). This prediction has been confirmed experimentally with high precision [13]. The result $\omega^2 = 3\omega_z^2$ was derived in [14–16], while the result $\omega^2 = 4\omega_z^2$ for the Tonks-Girardeau gas is simply understood by recalling that, in this limit, there is an exact mapping with the 1D ideal Fermi gas [10] where the excitation spectrum, in the presence of harmonic confinement, is $\omega = k\omega_z$. The same result has been recently derived in [17] using the mean field equations of Kolomeisky et al. [18].

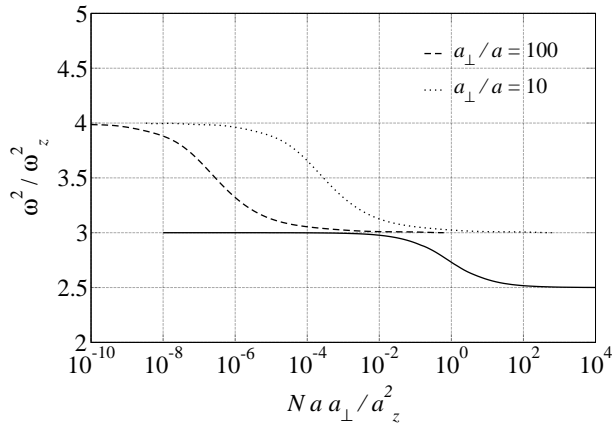


FIG. 1. Transition between the 1D mean field and the 3D cigar regimes: ω^2/ω_z^2 as a function of the parameter $N a a_\perp / a_z^2$ (full line). Transition to the Tonks-Girardeau regime for $a_\perp/a = 100$ (dashed line) and 10 (dotted line).

In order to evaluate the collective frequencies in the intermediate regimes where the hydrodynamic equations are not analytically soluble, we have developed a sum rule approach [11], based on the evaluation of the ratio $\hbar^2\omega^2 = m_1/m_{-1}$ between the energy weighted and inverse energy weighted sum rules. In the following we will limit the discussion to the lowest compression mode which is naturally excited by the operator $\sum_{i=1}^N z_i^2$. The energy weighted moment is given by $m_1 = (1/2)\langle [\sum_{i=1}^N z_i^2, [H, \sum_{i=1}^N z_i^2]] \rangle = (2N\hbar^2/m)\langle z^2 \rangle$ where $\langle z^2 \rangle = \int n_1(z) z^2 dz / N$ is the average square radius fixed by the ground state solution $n_1(z)$. The inverse energy weighted moment m_{-1} is related to the static polarizability α by $m_{-1} = (1/2)\alpha$. This is evaluated by adding the perturbation $-\epsilon z^2$ to the Hamiltonian and calculating the corresponding changes $\delta\langle z^2 \rangle$ of the expectation value of the square radius: $\alpha = N\delta\langle z^2 \rangle/\epsilon$. Adding the perturbation $-\epsilon z^2$ is equivalent to changing the frequency ω_z

of the harmonic confinement, so that the result for the collective frequency takes the compact form [19]

$$\omega^2 = -2 \frac{\langle z^2 \rangle}{d\langle z^2 \rangle/d\omega_z^2}. \quad (12)$$

We have calculated the ratio ω^2/ω_z^2 as a function of the dimensionless parameter $N a a_\perp / a_z^2$, thereby exploring the transition between the 3D cigar and the 1D mean field regimes. The results are reported in Fig. 1. For the experimental conditions of [1,2] where $N a a_\perp / a_z^2 = 0.24$ and 0.08, we predict $\omega^2/\omega_z^2 = 2.85$ and 2.91 respectively, confirming that those experiments are actually touching the transition between the two mean field regimes.

It is worth noticing that result (12) for the collective frequency, being based on general sum rule arguments, applies also to regimes beyond mean field where the density profile $n_1(z)$ and hence the value of $\langle z^2 \rangle$ cannot be evaluated starting from the Gross-Pitaevskii equation of state. Deviations from the mean field regime become important when the healing length $\xi = (8\pi a n)^{-1/2}$ is comparable to the average distance d between particles. In the presence of tight radial confinement one can use the relationship $n = n_1/\pi a_\perp^2$ between the 3D density evaluated at $r_\perp = 0$ and the 1D density $n_1 = \int n(r_\perp) d\vec{r}_\perp$. When a becomes smaller than d , one can write $d = 1/n_1$. One then obtains the result $\xi/d = \sqrt{a_\perp^2 n_1 / 8a}$ which becomes smaller and smaller as the 1D density decreases, thereby suggesting the occurrence of important deviations from the mean field behaviour for very dilute 1D samples. This should be contrasted with the 3D case where the mean field condition ($\xi > d$) is better and better satisfied as the density decreases. The combination $a_\perp^2 n_1 / a$ can then be used as an indicator of the applicability of the mean field approach. When its value becomes of the order of 1 or smaller, one enters a new regime characterized by important quantum correlations. The corresponding many-body problem was investigated by Lieb and Liniger [5] who considered 1D repulsive zero-range potentials of the form $g_{1D}\delta(z)$. The interaction parameter g_{1D} can be also written as $g_{1D} = \hbar^2/(ma_{1D})$ where a_{1D} is the one-dimensional scattering length [20]. By averaging the 3D interaction $4\pi^2\hbar^2 a \delta(\mathbf{r})/m$ over the radial density profile, one obtains the simple identification $a_{1D} = a_\perp^2/a$ [21]. The comparison with the expression for ξ/d shows that the deviations from the mean field increase by decreasing $a_{1D}n_1$.

In the Lieb-Liniger scenario the energy per particle $\epsilon(n_1)$, when expressed in units of the energy $\hbar^2/2ma_{1D}^2$, turns out to be a universal function of the dimensionless parameter $a_{1D}n_1$. Important limits are the high density limit $a_{1D}n_1 \gg 1$, where one finds $\epsilon(n_1) = \hbar^2 n_1 / ma_{1D}$ and hence the mean field result (7) for the chemical potential $\mu_{le} = \partial(n_1\epsilon(n_1))/\partial n_1$ (a part from the constant term arising from the radial external force). Note that under the assumption $a_\perp \ll a$, the two conditions

$an_1 \ll 1$ and $a_{1D}n_1 \gg 1$ required to realise the 1D mean field regime can be simultaneously satisfied. The other important limit is the low density Tonks-Girardeau limit $a_{1D}n_1 \ll 1$, where the chemical potential takes the value

$$\mu_{\ell e} = \pi^2 \hbar^2 n_1^2 / 2m. \quad (13)$$

Here the chemical potential no longer depends on the interaction coupling constant and reveals a typical Fermi-like behaviour [10].

In the presence of axial harmonic trapping the ground state density profile has been evaluated in [7] using the local density approximation (3). In this case the normalization condition $\int n_1(z) dz = N$ takes the form

$$\frac{Z a_{1D}}{a_z^2} \int_{-1}^1 \tilde{\mu}_{\ell e}^{-1} \left[\left(\frac{Z a_{1D}}{a_z^2} \right)^2 (1 - t^2) \right] dt = \frac{N a_{1D}^2}{a_z^2}. \quad (14)$$

Similarly to Eq.(8), we have introduced the radius Z at which the density vanishes and the inverse of the function $\tilde{\mu}_{\ell e}(n_{1D} a_{1D})$, where $\tilde{\mu}_{\ell e}$ is now the chemical potential expressed in units of $\hbar^2 / 2ma_{1D}^2$. Eq.(14) shows that the relevant combination of parameters is given by $N a_{1D}^2 / a_z^2$. This differs by the factor $(a_{\perp} / a)^3$ from the combination $N a a_{\perp}^2 / a_z^2$ characterizing the transition between the mean field regimes discussed in the first part of this work.

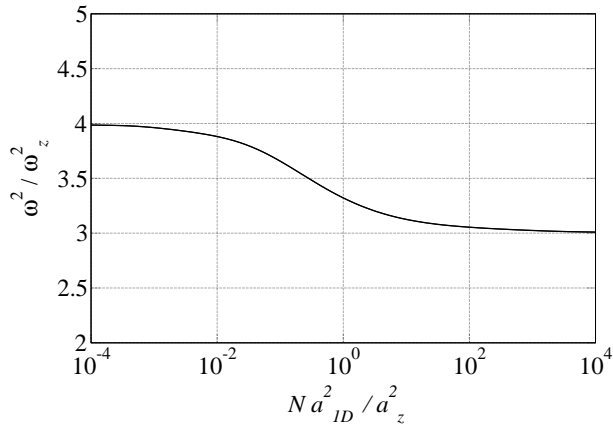


FIG. 2. Transition between the Tonks-Girardeau and the 1D mean field regimes: ω^2/ω_z^2 as a function of the parameter $N a_{1D}^2 / a_z^2$.

Analytic solutions are obtained in the two limits $N a_{1D}^2 / a_z^2 \gg 1$ and $N a_{1D}^2 / a_z^2 \ll 1$. In the first case, one recovers the mean field result (10). In the second one, we find the profile [8,18]

$$n_1(z) = \frac{\sqrt{2N}}{\pi a_z} \left(1 - \frac{z^2}{Z^2} \right)^{1/2}, \quad (15)$$

with $Z = \sqrt{2N} a_z$. In this case the applicability of the local density approximation simply requires $N \gg 1$.

By determining numerically the density profiles in the intermediate regimes we have calculated the frequency

of the lowest compressional mode using the sum rule formula (12). The results are reported in Fig. 2 as a function of $N a_{1D}^2 / a_z^2$. Fig. 1 shows the evolution of the collective frequency as a function of the parameter $N a a_{\perp} / a_z^2$ for two different choices of the ratio a_{\perp} / a . The corresponding curves reveal the transition between 3D cigar, 1D mean field and Tonks-Girardeau regimes. It is however worth pointing out that if the condition $a_{\perp} \gg a$ is not well satisfied, the 1D mean field regime is not clearly identifiable (see for example the dotted line in Fig. 1). In this case the determination of the collective frequencies requires the simultaneous inclusion of 3D and beyond mean field effects.

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